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Logarithmically slow onset of synchronization

Gil Benkö[1,2] **and Henrik Jeldtoft Jensen**[1,2]

[1] Institute for Mathematical Sciences, Imperial College London, 53 Prince's Gate, South Kensington Campus, SW7 2PG, London, UK
[2] Department of Mathematics, Imperial College London, South Kensington Campus, SW7 2AZ, London, UK

E-mail: g.benkoe@imperial.ac.uk and h.jensen@imperial.ac.uk

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Abstract

The transient of a synchronizing system is investigated, considering synchronization as a relaxation phenomenon. The stepwise establishment of synchronization is studied in the system of dynamically coupled maps introduced by Ito and Kaneko (2001 *Phys. Rev. Lett.* **88** 028701, 2003 *Phys. Rev.* E **67** 046226), where the plasticity of dynamical couplings might be relevant in the context of neuroscience. Logarithmically slow dynamics in the transient of a fully deterministic dynamical system are shown to occur.

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1. Introduction

Transients in the dynamics of complex systems and relaxation dynamics can inform on the underlying energy, fitness or cost landscape of a system (Wales 2004), and thus help to understand it better as a whole. Also, the transient state can be more relevant than the equilibrium state if most or more of the time is spent in the former.

The focus of the present paper is the systematic study of transients in the context of synchronization, a dynamical property of networks that is widely observed in fields such as optics, chemistry, biology and ecology, for instance in the brain (Gray *et al* 1989, Varela *et al* 2001) or in fireflies. Synchronization has been analysed for many physical systems (Pecora *et al* 1997, Pikovsky *et al* 2001b). The onset of synchronization is of general relevance. Similarity with relaxation, for example in spin glasses, or in superconductors, can be used to study synchronization. Transients can be used to probe the properties of synchronizing systems (Arenas *et al* 2006). However, there has been little research so far on synchronization as a relaxation phenomenon (Abramson 2000, Manrubia and Mikhailov 2000).

A specific form of relaxation dynamics is glassy dynamics, when the transient is extremely long (Jensen and Sibani 2007). Furthermore, in a number of systems with glassy dynamics the special case of logarithmically slow dynamics has been observed. However, so far all of these systems were stochastically driven (Anderson *et al* 2004, Sibani and Littlewood 1993, Sibani and Pedersen 1999). In the present paper, logarithmically slow dynamics in the transient of a fully deterministic dynamical system are shown to occur. In the first part of the paper the model is explained, a globally coupled map (GCM), with adaptive coupling which is inspired by the plasticity of synapses. Then the characteristics of its transients for a range of parameters are studied. Finally the transient is shown to be logarithmically slow for some parameters and that it can be explained by a simple model.

2. The Ito–Kaneko model

The Ito–Kaneko model (Ito and Kaneko 2001, 2003) is a GCM, a coupled simultaneous system of N logistic equations. The individual maps or units x^i are defined and coupled by

$$\begin{aligned} x^i_{t+1} &= (1-c)f\left(x^i_t\right) + c\sum_{j=1}^{N} w^{ij}_t f\left(x^j_t\right) \\ f(x) &= ax(1-x) \\ w^{ij}_{t+1} &= \frac{\left[1+\delta g\left(x^i_t, x^j_t\right)\right]w^{ij}_t}{\sum_{q=1}^{N}\left[1+\delta g\left(x^i_t, x^q_t\right)\right]w^{iq}_t} \\ g(x,y) &= 1-2|x-y|, \end{aligned} \tag{1}$$

where a is the logistic equation parameter and c is the coupling parameter. The coupling is further tuned by weights w^{ij}, which are dynamical variables as well. The function g, scaled by a parameter δ set to 0.1 here, defines a Hebbian update of the connection weights, by reinforcing the connections between similar units. This Hebbian plasticity of the couplings is inspired by the synaptic plasticity which enables nervous systems to learn (Antonov *et al* 2003).

The system exhibits three different long-term behaviours which can be classified into three phases, depending on the parameters. In the coherent phase C, all units synchronize, forming one synchronized cluster containing all units. In the ordered phase O, the set of units is partitioned into subsets or clusters C_k within which there is synchronization or which contain single units not synchronized with any other unit. Clusters of any size between 1 and N can be observed in this phase. In the following, in line with Ito and Kaneko (2001, 2003) and Manrubia and Mikhailov (2000), the number of parts in this partition will be called the cluster number. Finally, in the disordered phase D, no synchronization at all is achieved, forming a partition with a cluster number of value N. The phase diagram in figure 1(*a*) shows the boundaries between predominant phases in a part of the parameter space. Ito and Kaneko have performed a bifurcation analysis for the dynamics of an isolated synchronizing pair around the boundary O/D (Ito and Kaneko 2003).

3. Synchronization transients

It has been found previously that in a static links version of the Ito–Kaneko model its transient is exponential in the ordered regime O and that its length diverges when the border O/D between the ordered and disordered regimes is approached (Abramson 2000, Manrubia and Mikhailov 2000). Stretched exponential decay of correlation functions has been found in a similar system (Katzav and Cugliandolo 2005).

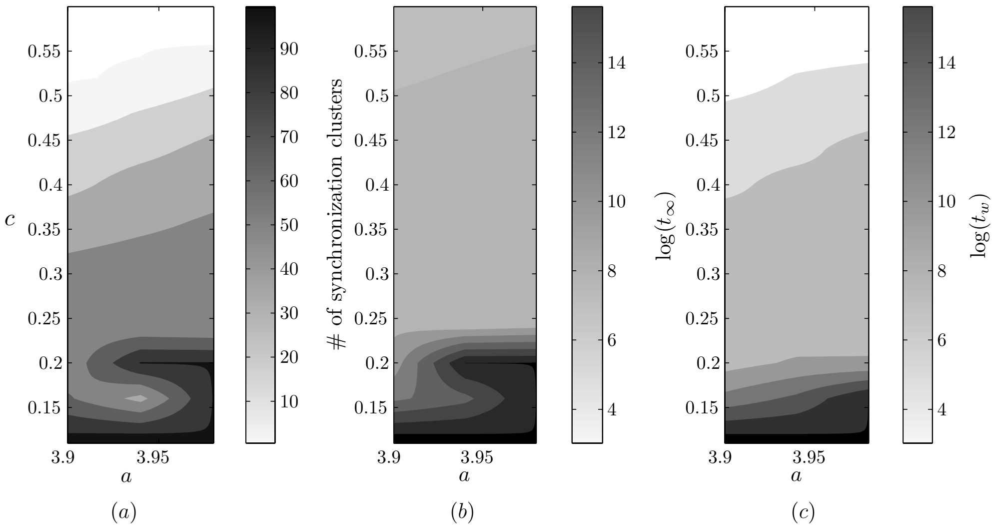


Figure 1. (*a*) Part of the phase diagram of the Ito–Kaneko model. (*b*) Length of the transient t_∞. (*c*) Time t_w until the variation of weights falls below 0.1% (all averaged over 25 realizations).

In the following, the transient is studied in detail, focusing on the way events during the transient are simulated and detected. The system (1) was simulated numerically, each initial x^i being randomly chosen in the interval [0, 1]. The initial w^{ij} are set to $1/(N-1)$. A higher precision simulation method similar to that of Pikovsky *et al* (2001a) was used in order to avoid synchronization artefacts due to limited numerical precision. Two units x^i and x^j were considered to be synchronized if $|x^i - x^j| < 10^{-270}$. Due to the resulting increased computation times the focus was on a part of the parameter space, shown in figure 1. During each run of a simulation, the time steps t_k at which two units synchronized were recorded. These t_k will be called synchronization events in the rest of the present paper. As the successive synchronization is analysed in analogy to the investigation of avalanches and quakes in Anderson *et al* (2004), where a set of events is considered as one quake, here any synchronization events less than $\Delta t = 10$ time steps apart are considered together as a single synchronization event.

In figure 4(*a*) an example of the time evolution of the number of synchronized clusters for a single simulation run is shown. Figure 4(*b*) is discussed later. The number of clusters in figure 4(*a*) drops stepwise from $N = 100$ during a long transient. The number of synchronization events until time step t is noted $N_{\mathrm{sync}}(t)$. If, after a synchronization event at time step t_k, no further synchronization events happened for a duration of $4t_k$, the simulation was stopped. The synchronization transient is defined as the time from the beginning of the simulation until the last synchronization event.

The dependence of the transient on the parameters a and c is shown in figures 1(*b*) and (*c*). For most parameters, the system relaxes fast, i.e. is non-glassy. Only for $c < 0.2$, the main feature is that the length of the transient, and also the time until the connection weights w^{ij} stabilize, increases considerably as the overall coupling c decreases, i.e. it is glassy. Figure 2 shows again in detail how the length of transient diverges as the coupling c diminishes, with the parameter of the logistic function $a = 3.9$ constant. This intuitively makes sense as the system changes from a regime with some synchronization (ordered O) to a regime with no synchronization at all (disordered D). As c decreases, the units x_i influence each other less and

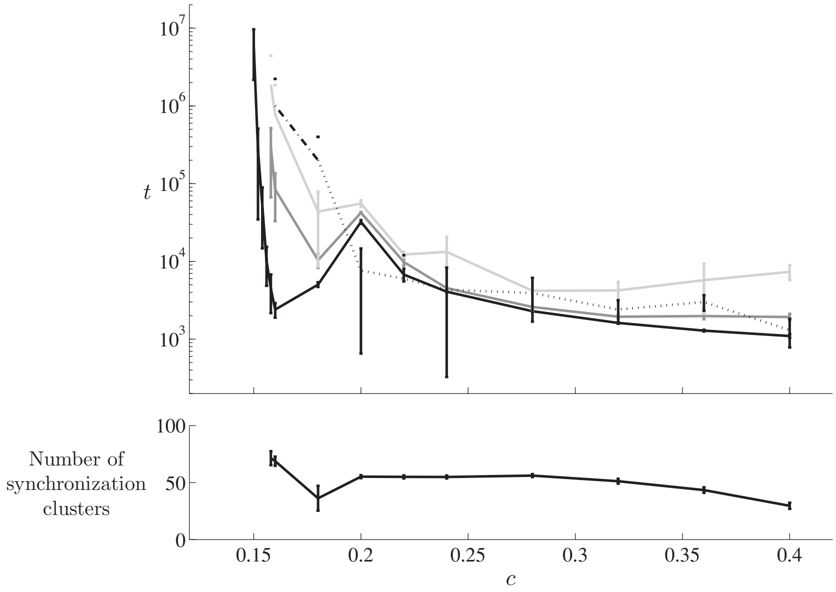


Figure 2. Top: divergence of transient length as the border to the disordered region is approached, with $a = 3.9$. Shown are the time of the first synchronization event t_1 (solid black) of the 25th synchronization event t_{25} (dark grey) and of the last synchronization events t_∞ (light grey), and t_w (dotted black when w converges in all realizations, dash-dotted otherwise). Bottom: the corresponding number of synchronization clusters (averaged over 25 simulation runs).

synchronize less easily, which leads to the observed transition from synchronization O to no synchronization D. For the rest of the present paper, a is fixed to 3.9.

The divergence of the transient corresponds to the border between the ordered and disordered regimes. Ito and Kaneko have derived a theoretical value for the coupling $c_{O/D}$ at which the length of the transient diverges for a system in which the weights fulfil the condition $w^{iq} = w^{jq}$ as the units x^i and x^j synchronize (Ito and Kaneko 2003). As shown in figures 1(*c*) and 2, the weights do not always stabilize in the present system, especially not as c decreases, and might not fulfil the above condition. Figure 2 shows that in a part of the realizations for c near $c_{O/D}$ (dash-dotted), the weights do not stabilize at all within the simulation run, even though there is synchronization. It is nevertheless interesting to calculate $c_{O/D}$ for a system in which the weights fulfil the condition $w^{iq} = w^{jq}$ and compare it to the value for $c_{O/D}$ obtained from simulation.

The border between the ordered and disordered regimes can be calculated by studying the stability of the ordered state, using the transversal Lyapunov exponent. At the considered border only synchronization clusters of size up to 2, i.e. only isolated pairs of synchronized units, are formed. The transversal Lyapunov exponent for a system in which $w^{iq} = w^{jq}$ is then (Ito and Kaneko 2001, 2003)

$$\Lambda_\perp = \ln|1 - 2c| + \Lambda_f, \tag{2}$$

where Λ_f is the Lyapunov exponent of $f(x)$, the logistic function. For $a = 3.9$, the Lyapunov exponent of the logistic function calculated by simulation is $\Lambda_f \approx 0.73$ and a theoretical value $c \approx 0.26$ is obtained. In the simulations of (1), which has dynamical weights, $c_{O/D} \approx 0.15$. Thus, there is indeed a difference between the analytical result and the simulated result. It is possible that the very adaptive nature of a system in which the weights do not stabilize and

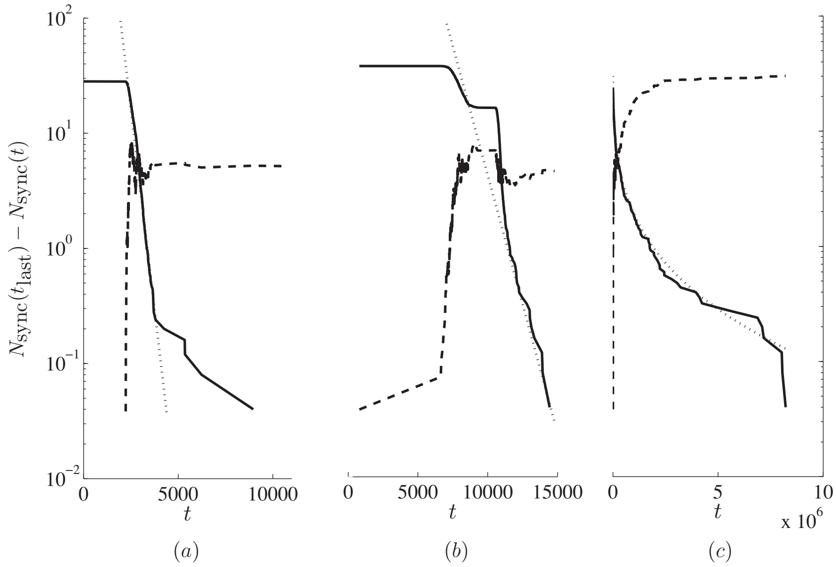


Figure 3. Time dependence of $N_{\mathrm{sync}}(t_\infty) - N_{\mathrm{sync}}(t)$ averaged over 25 simulation runs (solid) with variance (dashed) and fitted exponential or stretched exponential of the form $\exp(-t^{0.28})$ (dotted), $c = 0.28, 0.22, 0.158$ (from left to right).

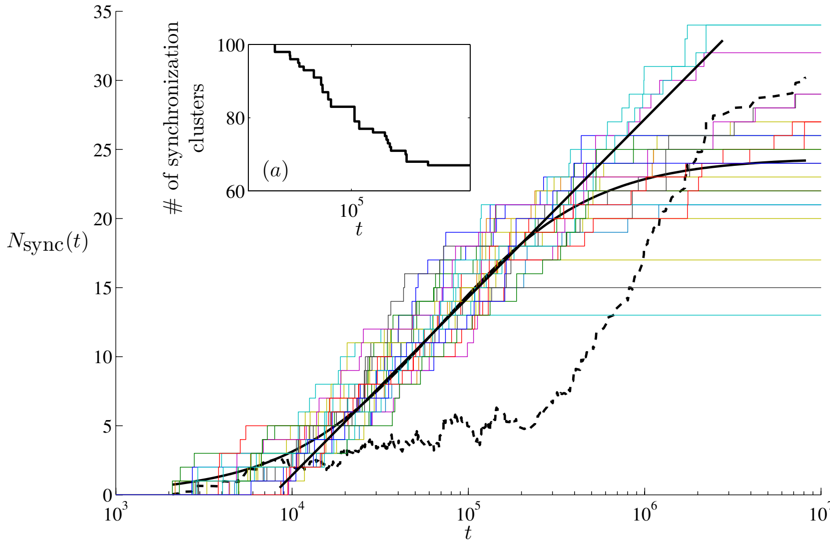
might not fulfil the condition $w^{iq} = w^{jq}$ results in less overall connectivity being needed to achieve synchronization.

The transient was studied in a logarithmic timescale for c values close to and far from $c_{O/D}$. In figure 3, $N_{\mathrm{sync}}(t_\infty) - N_{\mathrm{sync}}(t)$ is plotted against time in order to detect exponential transients of the form $N_{\mathrm{sync}}(t_\infty) - N_{\mathrm{sync}}(t) \propto \mathrm{e}^{-\beta t}$, as described in Abramson (2000), Manrubia and Mikhailov (2000). For $c = 0.28$, shown in figure 3(*a*), the transient is indeed exponential. However, as c decreases, the transient turns more irregular, see figure 3(*b*) ($c = 0.22$). Eventually, for $c = 0.158$, very close to $c_{O/D}$, the transient is extremely long and settles into the stretched exponential shape shown in figure 3(*c*). Stretched exponential relaxation has been observed in related systems (Katzav and Cugliandolo 2005). Also the evolution of the variance of the considered quantity is different depending on whether c is away or very close to the border. The difference in the shape of the transient and its variance indicates that a different process is underlying the system at the border O/D. In the following the transient statistics at the border O/D are investigated further.

4. Logarithmically slow transients

The dynamics at the border O/D is unusual, characterized by an extremely long transient, and thus of particular interest. Also, this border separates ordered from disordered behaviour and is interesting because of the relevance of computation at the edge of chaos in neural nets (Natschläger *et al* 2005).

Extremely long transients have also been observed in other many component systems with glassy dynamics (Jensen and Sibani 2007). There, the time span needed to reach a steady, time-independent state is often far beyond experimentally accessible time scales. For example, when melted alloys are cooled down they typically retain the amorphous arrangement



(b)

Figure 4. Synchronization transient near the border O/D, with $c = 0.158$. (a—inset) Time dependence of the number of synchronization clusters for a single simulation run. (*b*) Time dependence of the number of synchronization events for 25 simulation runs (thin lines) and corresponding variance (dashed), a linear fit and a fit of the simple model described by (5) (thick lines).

(This figure is in colour only in the electronic version)

characteristic of the liquid high-temperature phase, while the molecular mobility decreases many orders of magnitude, rendering it near impossible to reach thermodynamic equilibrium. However, over short time scales, the system properties may appear to be time independent as in thermal equilibrium. Only when several orders of magnitude of time scales are covered, the slow change of macroscopic characteristic properties with time can be resolved directly.

A superposition of 25 simulations runs for $c = 0.158$ close to $c_{O/D}$ covering the whole very long transient up to $t = 10^7$ is shown in figure 4. Due to computational complexity no value closer to $c_{O/D}$ was used. Interestingly, the transient is linear on a logarithmic timescale for intermediate times. There also seem to be two types of behaviours at the end of the transient, the 25 simulations split into two groups there. There might be two different groups of attractors, preliminary calculations yield a similar split for different system sizes (not shown). In the following an approach to modelling the transient is described.

A simple model would be to first assume that the rate of synchronization events is proportional to the rate at which any two units have close values by chance, i.e. 'collide'. However, at low overall coupling c, units x^i have been observed to only synchronize with a single additional unit and form pairs (Ito and Kaneko 2003). This observation can be studied analytically by calculating the stability of a synchronized cluster, i.e. the evolution of the synchronization error. If x^i and x^j are part of a cluster, the synchronization error $x^\Delta = x^i - x^j$ evolves as

$$x^\Delta_{t+1} \approx \left(1 - c\left(1 + w^{ij}_t\right)\right) f'\left(x^i_t\right) x^\Delta_t + c \sum_{q\neq i,j} \left(w^{iq}_t - w^{jq}_t\right) f\left(x^q_t\right). \tag{3}$$

The connection strength within a synchronized cluster is $w^{ij} \approx 1/(N_C - 1)$, where N_C is the size of the cluster (Ito and Kaneko 2003). This indicates that if two comparable clusters of different size form, the growth of the synchronization error of the bigger cluster will be larger and the bigger cluster will be less stable. In the present case, when c is very low, only clusters of size 2, i.e. pairs, seem to be stable.

Thus, units that are already paired up are assumed not to be available anymore to synchronize with other units. So simplifying further, if the number of synchronization events is noted $N_{\text{sync}}(t)$, and the number of unsynchronized units is approximately $N - N_{\text{sync}}(t)$,

$$\frac{\mathrm{d}N_{\text{sync}}(t)}{\mathrm{d}t} = k(N - N_{\text{sync}}(t))^2 \tag{4}$$

and

$$N_{\text{sync}}(t) = N - \frac{1}{\frac{1}{N} + kt}. \tag{5}$$

Assuming furthermore that only an effectual subset of the N units can actually synchronize due to the low overall coupling c, N is replaced by an effectual, i.e. actually operating number of units N_{eff} in the above formula. The use of an effectual number of units N_{eff} might be a result of the weights not stabilizing at low c as mentioned above. Synchronized pairs might still affect unsynchronized units by being weakly connected to them and might render their synchronization less probable.

The predicted $N_{\text{sync}}(t)$ agrees well with the simulated data for $N_{\text{eff}} = 24.5$ and $k = 6 \times 10^{-7}$, see figure 4(*b*). However, it is not possible to find a reasonable fit of the simulated $N_{\text{sync}}(t)$ to (5) for higher values of c. As explained above, only pairs were assumed and observed to form at very low c. For higher values of c, the simple model which uses the assumption that only pairs can be formed and which leads to (5) does not hold anymore, and indeed, the observed transient has a different shape.

Furthermore $N_{\text{sync}}(t)$ seems to be linear on a logarithmic timescale for intermediate times. This might be related to the log-time dependence that has been previously obtained for non-homogeneous Poisson processes in logarithmic time (log-Poisson). Log-Poisson processes have been observed in the NK model of evolution (Sibani and Pedersen 1999), charge-density waves (Sibani and Littlewood 1993), and further in spin glasses, supercooled magnet relaxation and the tangled nature evolution model (Anderson *et al* 2004). The hypothesized mechanism behind these processes is inspired by intermittency studies of fluctuations in glassy systems that have demonstrated that large intermittent fluctuations are responsible for the deviations from equilibrium statistics (Buisson *et al* 2003). It was suggested that abrupt and irreversible moves from one metastable configuration to another, the so-called quakes, are a result of record-sized fluctuations. The assumption that the metastable attractors typically selected by the glassy dynamics have marginally increasing stability means that a fluctuation bigger than any previously occurred fluctuation, i.e. a record-sized fluctuation, can induce a quake (Sibani and Dall 2003, Sibani and Littlewood 1993). Quakes lead to entrenchment into gradually more stable configurations, and carry the average drift of the dynamics. The quakes have a similar effect on a logarithmic time scale, which might be modelled by a Poisson process in logarithmic time.

While in a Poisson process the probabilities for events t_k are characterized by van Kampen (2007):

$$P[N(t+\tau) - N(t) = k] = \frac{\mathrm{e}^{-\lambda\tau}(\lambda\tau)^k}{k!} \tag{6}$$

and

$$P[t_k - t_{k-1} < x] = 1 - \mathrm{e}^{-\lambda x}, \tag{7}$$

in analogy, in a log-Poisson process (Sibani and Dall 2003):

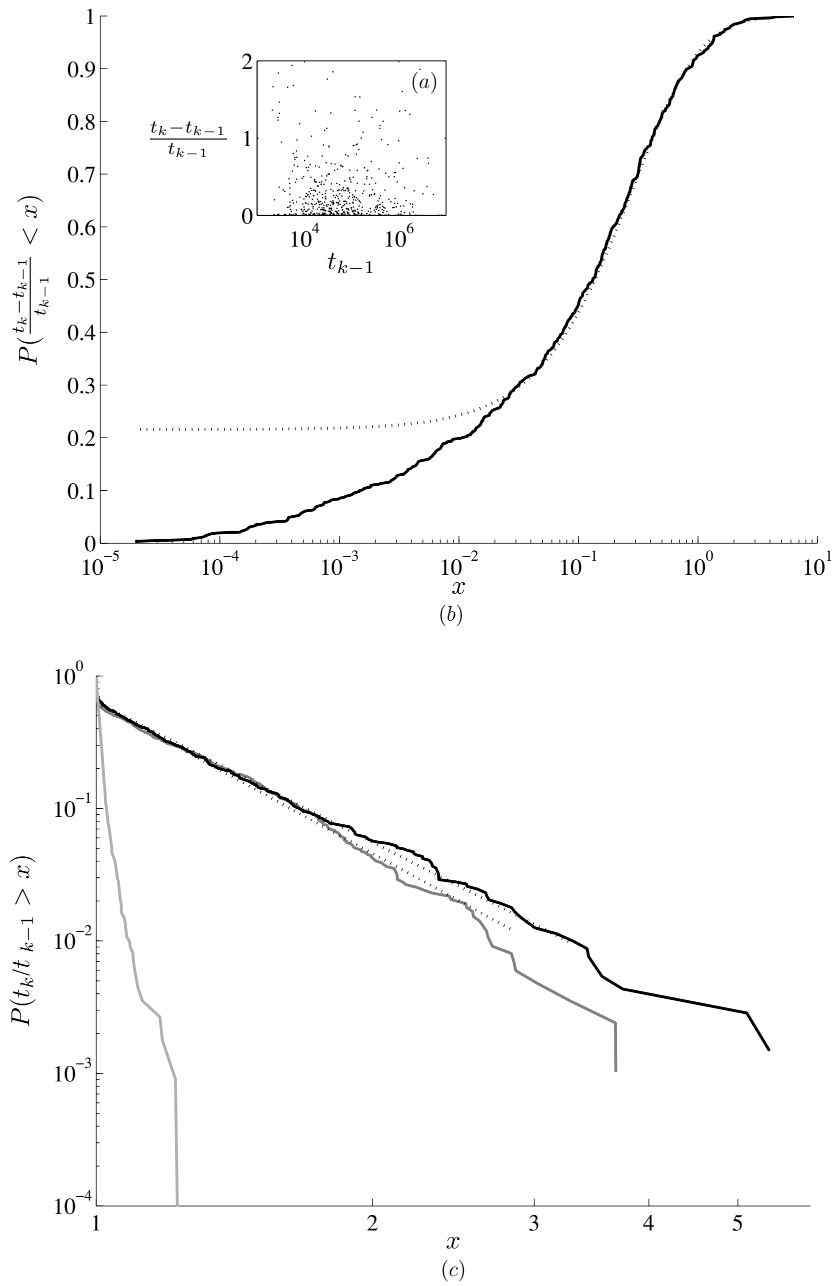
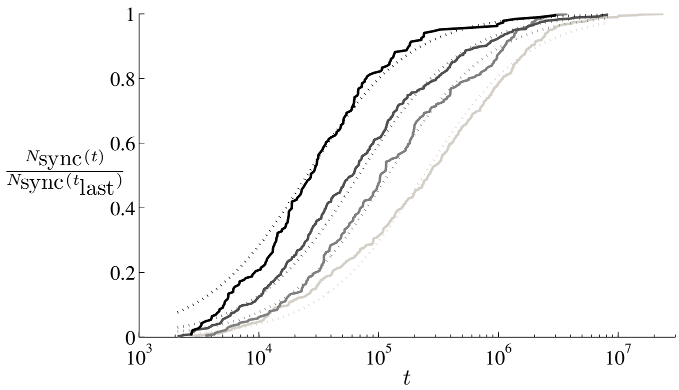
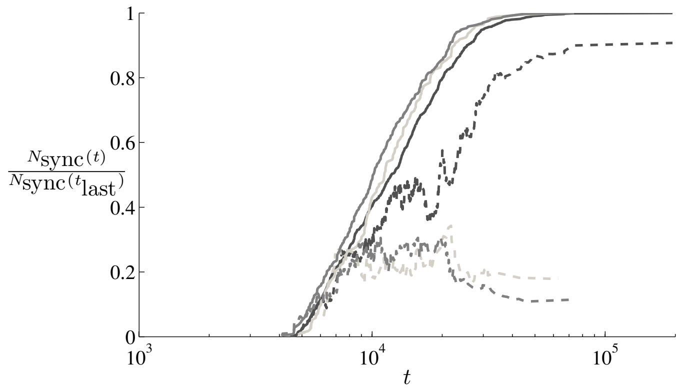


Figure 5. (a—inset) Distribution, (*b*) cumulative distribution of $(t_k - t_{k-1})/t_{k-1}$ for $c = 0.158$ (solid) and theoretic distribution for a log-Poisson process (dotted). (*c*) Cumulative distribution of t_k/t_{k-1} close to the border O/D with $c = 0.2$ (light grey), 0.16 (dark grey) and 0.158 (black), and fitted theoretic distribution for a log-Poisson process (dotted).



(*a*)



(*b*)

Figure 6. System size dependence of the transient. (*a*) Transients for $c = 0.158$ and $N = 25, 50, 100$ and 200 (solid grey to black) and corresponding theoretical transients using (5), $N_{\rm eff} = 7, 14, 24.5$ and 61 and $k = 6 \times 10^{-7}$ (dotted). (*b*) Transients for $c = 0.18$ and $N = 25, 50$ and 100 (solid) and variance (dashed).

$$P[N(t+\tau) - N(t) = k] = \frac{1}{k!}\left(\frac{t+\tau}{t}\right)^{-\lambda}\left(\lambda\frac{t+\tau}{t}\right)^{k} \tag{8}$$

$$P[\ln(t_k/t_{k-1}) < x] = 1 - \mathrm{e}^{-\lambda x}. \tag{9}$$

In both cases, it is easy to show that the PDF of the number of events $N(t)$ and its variance are equal. Thus, in figure 3 $N_{\rm sync}(t_\infty) - N_{\rm sync}(t)$ and its variance should be symmetric, which is observed indeed only up to some error on the border O/D (figure 3(*c*)), while away from the border (figures 3(*a*) and (*b*)) $N_{\rm sync}(t_\infty) - N_{\rm sync}(t)$ and its variance seem unrelated.

Thus, the process described by (5) seems to be an alternative way to obtain log-time dependence in an intermediate time regime. This finding can be further corroborated by studying the cumulative distribution of $(t_k - t_{k-1})/t_{k-1}$, see figure 5(*b*). For a static Poisson process, this distribution would be a step function. Instead, it agrees well with the theoretic distribution for a log-Poisson process, $P((t_k - t_{k-1})/t_{k-1} < x) = 1 - (1+x)^{-\lambda}$ with $\lambda = 3.5$. The support for the distribution at low $(t_k - t_{k-1})/t_{k-1}$ comes from the initial phase of the transient, where the initial conditions dominate, which might explain the discrepancy between the theoretical and simulated distribution there. Also, as shown in figure 5(*a*), the value of $(t_k - t_{k-1})/t_{k-1}$ stays within a relatively small range over a range of orders of magnitude of t_k, while it would quickly drop to zero in a static Poisson process. In figure 5(*c*) the cumulative distribution of t_k/t_{k-1} is shown. This distribution also agrees well with the theoretical cumulative distribution (9) for a log-Poisson process with $\lambda = 3.5$ and seems to corroborate that the synchronization process in this model is an alternative way to obtain logarithmically slow relaxation dynamics.

The size of a system can be relevant to the shape of its transient. A superposition of the transients for different system sizes, normalized by the final number of synchronization events, and for $c = 0.158$ close to $c_{O/D}$ and $c = 0.18$, is shown in figures 6(*a*) and (*b*). Due to computational complexity, the sampling size is small, especially for $N = 200$. The simulations show that the system size dependence close to $c_{O/D}$ seems to be a shifting of the transient to faster synchronization. The shifting is close to what (5) suggests. Also, for $c = 0.18$, further away from $c_{O/D}$, the transients are almost equivalent for different system sizes when normalized by the final number of synchronization events. Thus, also here the system synchronizes faster for bigger system sizes. The faster synchronization for increasing system size can be explained similarly to (5): since the values of the units x^i are constrained to the interval [0, 1] of the logistic map, increasing the system size increases the number of units in [0, 1] and therefore increases the number of collisions, i.e. the speed of synchronization.

5. Conclusion

In conclusion, synchronization was investigated as a relaxation phenomenon. The transient of synchronization in a coupled map model was found to drastically depend on the amount of overall coupling. For an overall coupling at the border to disorder, the transient was found to be logarithmically slow at intermediate times. This behaviour has been found in other systems exhibiting record statistics, where the dynamics is termed log-Poisson and arises because record fluctuations become increasingly rare. In the model of the present system the dynamics may arise by an alternative way: it can be simply explained by (for intermediate times) it getting more and more difficult to find a new synchronization partner and actually synchronize as the system ages. Interestingly, this kind of dynamics has been found otherwise in noise-driven systems. The coupled map model used in the present paper has no noise term; however, it uses the logistic map, which has been studied as a noise generator since Ulam and von Neumann (1947). Thus, the use of the logistic map might be an ingredient for the appearance of logarithmically slow dynamics. In a similar system it was shown that a coupled map lattice is equivalent to a system of stochastic PDE (Katzav and Cugliandolo 2005).

One example for the relevance of the synchronization observed in GCM is neural networks. The speed of synchronization in neural networks is important (Kopell and Ermentrout 2004), as it presumably is related to the response time of the brain. Namely, if it is assumed, along with Varela *et al* (2001), that the state of the brain is given by its state of synchronization, then the time to swap from one synchronization pattern to another seems to determine how fast the brain can react. Also, the behaviour at the edge between partial synchronization and disorder is especially interesting because of the relevance of computation at the edge of chaos (Natschläger *et al* 2005).

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